Planck's law and Planck's radiation formula

In 1900, Planck introduced entirely new ideas to explain the distribution of energy among the various wavelengths of cavity (black body) radiation. In classical theory, it was assumed that the energy changes of radiation take place continuously. This led to self contradictory results. Therefore, Planck argued that the classical idea of continuity of action might be wrong. Instead of it he proposed that the energy change could take place only discontinuously and discretely always ~~as~~ as integral multiples of a small unit of energy called the "quantum" or "photon".

According to classical theory, energy changes of radiation take place continuously but it could not explain experimentally observed distribution of energy in the spectrum of black body. ~~Planck, therefore~~ In 1900, Planck introduced or proposed an important hypothesis known as planck's hypothesis. According to it "A black body radiation chamber is filled up not only with radiation, but also with simple harmonic oscillators or resonators of the molecular dimensions and the exchange of energy among the various wave lengths of black body radiation does not take place continuously but discontinuously and discretely as an integral multiple of small unit of energy called the "quantum" or "photon." ~~Each photon has an energy~~

planck assumed that the energy of a

photon is proportional to the frequency of radiation i.e $E \propto \nu$
or $E = h\nu$ where
h is planck's constant and its value is 6.62×10^{-34} joule.sec.

Planck's radiation formula

Planck found an empirical formula to explain the experimentally observed distribution of energy in the spectrum of black body. The formula may be deduced using the following postulates —

I. A black body radiation chamber is filled up not only with radiations but also with simple harmonic oscillators or resonators of the molecular dimensions, which can not have any value of energy but only energy given by

$$E = nh\nu$$

Where ν is the frequency of the oscillator, h is the planck's constant and n is an integer equal to 0, 1, 2, 3, ...

II. The oscillators can not radiate or absorb energy continuously but an oscillator of frequency ν can only radiate or absorb energy in units or quanta of magnitude $h\nu$. This assumption is the most revolutionary in character.

In simple words this states that the exchange of energy between radiation and matter can not take place continuously but are limited to discrete set of values $0, h\nu, 2h\nu, \ldots\ldots n h\nu$ i.e in multiple of some small unit called the quantum.

Let $\bar{E}$ be the average energy of an oscillator of frequency ν. If N be the total number of oscillators and E be their total energy then average energy of an oscillator

$$\bar{E} = \frac{E}{N} \quad \text{———} \quad (1)$$

Let $N_0, N_1, N_2, N_3, \ldots\ldots N_n, \ldots$ be the oscillators having energies $0, h\nu, 2h\nu, \ldots, nh\nu$ ------ respectively. The relative probability that an oscillator has the energy $h\nu$ at temperature T is given by the Boltzmann factor $e^{-h\nu/KT}$, where K is the Boltzmann's constant. Thus we have

$$N_n = N_0 e^{-nh\nu/kT}$$

Total No. of oscillators is

$$N = N_0 + N_1 + N_2 + N_3 + \ldots + N_n + \ldots$$

$$= N_0 + N_0 e^{-h\nu/KT} + N_0 e^{-2h\nu/KT} + N_0 e^{-3h\nu/KT} + \ldots$$

$$= N_0 \left\{ 1 + e^{-h\nu/KT} + e^{-2h\nu/KT} + e^{-3h\nu/KT} + \ldots \right\}$$

[sum of G.P]

$$= N_0 \left\{ \frac{1}{1 - e^{-h\nu/KT}} \right\}$$

$$\text{or } N = \frac{N_0}{1 - e^{-h\nu/KT}} \quad \text{———} \quad (2)$$

Total energy of the oscillators is given by

$$E = N_0 \times 0 + N_1 \times h\nu + N_2 \times 2h\nu + N_3 \times 3h\nu + \ldots\ldots$$

$$= N_0 e^{-h\nu/kT} \times h\nu \left[\{1 - e^{-h\nu/kT}\}^{-2} \right]$$

$$\left[\because 1 + 2x + 3x^2 + \dots + rx^{r-1} + \dots = \frac{1}{(1-x)^2} \right]$$

$$\therefore E = N_0 e^{-h\nu/kT} \cdot \frac{h\nu}{(1 - e^{-h\nu/kT})^2} \quad \text{--- (3)}$$

Putting the value of (2) and (3) in (1)

$$\bar{E} = \frac{E}{N} = \frac{N_0 e^{-h\nu/kT} \cdot \dfrac{h\nu}{(1 - e^{-h\nu/kT})^2}}{\dfrac{N_0}{1 - e^{-h\nu/kT}}}$$

$$= \frac{\cancel{N_0} e^{-h\nu/kT} \cdot h\nu}{\cancel{(1 - e^{-h\nu/kT})^2}} \times \frac{(1 - e^{-h\nu/kT})}{\cancel{N_0}}$$

$$= \frac{h\nu \cdot e^{-h\nu/kT}}{1 - e^{-h\nu/kT}}$$

$$= \frac{h\nu \, \cancel{e^{-h\nu/kT}}}{(e^{h\nu/kT} - 1) \, \cancel{e^{-h\nu/kT}}}$$

$$\bar{E} = \frac{h\nu}{(e^{h\nu/kT} - 1)} \quad \text{--- (4)}$$

The ~~no~~ modes of vibration i.e. the no. of oscillators per unit volume in the frequency range ν and $\nu + d\nu$ is given by

$$\frac{8\pi\nu^2}{c^3} d\nu$$

The energy density or density of radiation

$E_\nu d\nu$ of frequencies between ν and $\nu+d\nu$ is related to the average energy of an oscillator emitting ν - radiation by

$$E_\nu d\nu = \frac{8\pi\nu^2}{c^3} d\nu \, \bar{E} \quad \text{—— (5)}$$

Putting (4) in (5) we get

$$E_\nu d\nu = \frac{8\pi\nu^2}{c^3} d\nu \cdot \frac{h\nu}{(e^{h\nu/KT} - 1)}$$

$$E_\nu d\nu = \frac{8\pi h}{c^3} \cdot \frac{\nu^3 d\nu}{(e^{h\nu/KT} - 1)} \quad \text{—— (6)}$$

This is planck's radiation formula in terms of frequency ν.

Now $\nu = \frac{c}{\lambda}$

$$d\nu = -\frac{c}{\lambda^2} d\lambda$$

Again it has been found that an increase in frequency corresponds to a decrease in wavelength.

$$E_\lambda d\lambda = -E_\nu d\nu$$

$$E_\lambda d\lambda = -\frac{8\pi h}{c^3} \cdot \frac{\left(\frac{c}{\lambda}\right)^3 \cdot \left(-\frac{c}{\lambda^2} d\lambda\right)}{e^{hc/KT\lambda} - 1}$$

or $$E_\lambda d\lambda = \frac{8\pi ch}{\lambda^5} \cdot \frac{1}{e^{hc/KT\lambda} - 1} \cdot d\lambda \quad \text{—— (7)}$$

This is planck's radiation formula in terms of wavelength.

Special Cases

(i) when λ is very small,

$$e^{hc/\lambda KT} >> 1$$

equation (7) reduces to

$$E_\lambda d\lambda = \frac{8\pi ch}{\lambda^5} \cdot \frac{1}{e^{hc/KT\lambda}} d\lambda$$

$$= \frac{8\pi ch}{\lambda^5} \cdot e^{-hc/KT\lambda} d\lambda$$

Putting $8\pi hc = A$ and $\frac{hc}{K} = B$, we get

$$E_\lambda d\lambda = \frac{A}{\lambda^5} e^{-B/\lambda T} d\lambda$$

$$\boxed{E_\lambda d\lambda = A\lambda^{-5} e^{-B/\lambda T} d\lambda}$$

This is Wein's law which agrees with experiments at short wavelength only.

(ii) when λ is large,

$$e^{hc/\lambda KT} \simeq 1 + \frac{hc}{\lambda KT}$$

Equation (7) reduces to

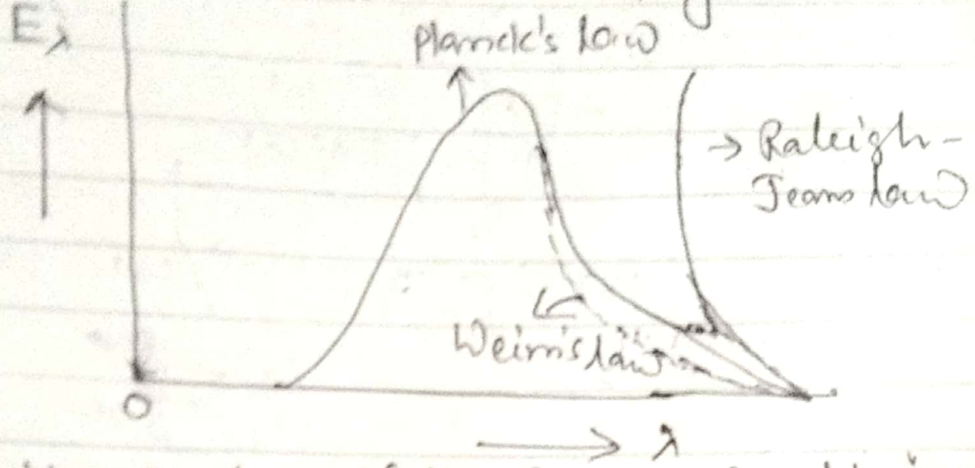
$$E_\lambda d\lambda = \frac{8\pi ch}{\lambda^5} \cdot \frac{1}{\cancel{1} + \frac{hc}{\lambda KT} - \cancel{1}} \cdot d\lambda$$

$$\boxed{E_\lambda d\lambda = \frac{8\pi KT}{\lambda^4} d\lambda}$$

This is Raleigh - Jeans law which agrees with experiment at long wavelengths only.

Comparison with experimental results

Rubens and Michel verified Planck's formula experimentally using a prism spectrometer with four radiators and obtained the curves as shown in the fig.



From the nature of the curves so obtained we find that for shorter wavelengths, Planck's law reduces to Wein's law. For long wavelengths, and high temp, this law reduces to Raleigh-Jeans law. Thus it can be said that Planck's law is a general law ~~and~~ while Raleigh-Jeans law are the special cases of it. Thus we find that Planck's formula for the energy distribution in thermal spectrum is applicable for all wavelengths.

Deduction of Stefan's law from Planck's radiation formula.

The Planck's radiation formula in terms frequency is

$$E_\nu d\nu = \frac{8\pi h}{c^3} \cdot \frac{\nu^3 d\nu}{e^{h\nu/kT} - 1}$$

(from eqn 6)

Hence the total radiant energy over all frequencies is

$$E = \int_0^\infty E_\nu \, d\nu$$

$$= \frac{8\pi h}{c^3}\int_0^\infty \frac{\nu^3 d\nu}{e^{h\nu/kT}-1}$$

Putting $x = h\nu/kT$

$\therefore \nu = \left(\frac{kT}{h}\right)x$

$\therefore d\nu = \left(\frac{kT}{h}\right)dx$

$\therefore \nu^3 d\nu = \left(\frac{kT}{h}\right)^3 x^3 \cdot \left(\frac{kT}{h}\right)dx$

$= \left(\frac{k}{h}\right)^4 T^4 x^3 dx$

Hence from above

$$E = \frac{8\pi h}{c^3}\cdot\left(\frac{k}{h}\right)^4 T^4 \cdot \int_0^\infty \frac{x^3 dx}{e^x - 1}$$

$$\int_0^\infty \frac{x^3 dx}{e^x - 1} = \frac{\pi^4}{15} \quad \text{(Calculated value)}$$

$$\therefore E = \frac{8\pi^5 k^4}{15c^3 h^3} T^4$$

$$\boxed{E = \sigma T^4}, \quad \text{Where } \sigma = \frac{15\pi^5 k^4}{15 c^3 h^3}$$

This is Stefan's law. $=$ a Constant Called Stefan Con

(ii) Wien's displacement law

From Planck's radiation formula

$$E_\lambda = \frac{8\pi ch}{\lambda^5} \cdot \frac{1}{e^{hc/\lambda KT} - 1}$$

$$\text{or } E_\lambda = 8\pi ch\, \lambda^{-5} \left(e^{hc/\lambda KT} - 1\right)^{-1}$$

The wavelength at which the spectral radiancy is maximum,

$$\frac{dE_\lambda}{d\lambda} = 0$$

$$8\pi ch\left[-5\lambda^{-6}\left(e^{hc/\lambda KT}-1\right)^{-1} + \lambda^{-5}(-1)\left(e^{hc/\lambda KT}-1\right)^{-2} \times e^{hc/\lambda KT} \times -\frac{hc}{\lambda^2 KT}\right] = 0$$

$$\left[-5(\lambda^{-6})\left(e^{hc/\lambda KT}-1\right) + \lambda^{-5}(-1)\left(e^{hc/\lambda KT}-1\right)^{-2} \times e^{hc/\lambda KT} \times -\frac{hc}{\lambda^2 KT}\right] = 0$$

$$\text{or } \lambda^{-6}\left(e^{hc/\lambda KT}-1\right)\left[-5 + \left(e^{hc/\lambda KT}-1\right)^{-1} e^{hc/\lambda KT} \times \frac{hc}{\lambda KT}\right] = 0$$

$$\frac{\left(e^{hc/\lambda KT}-1\right)^{-1}}{\lambda^6} \neq 0$$

$$-5 + \frac{1}{e^{hc/\lambda KT}-1} \times e^{hc/\lambda KT} \times \left(\frac{hc}{\lambda KT}\right) = 0$$

$$\text{or } 5 = \left(\frac{hc}{\lambda KT}\right) \frac{e^{hc/\lambda KT}}{e^{hc/\lambda KT}-1}$$

Putting $\frac{hc}{\lambda KT} = x$

$$5 = \frac{x e^x}{e^x - 1} = \frac{x}{1-e^{-x}}$$

$$5(1-e^{-x}) = x$$

$$5 - 5e^{-x} = x$$

$$1 - e^{-x} = x/5$$

$$\frac{x}{5} + e^{-x} - 1 = 0$$

This equation has a single root given by $x = 4.965$. and there x must be a constant.

i.e $\frac{hc}{\lambda KT} = 4.965$

Therefore, the wavelength λ_m at which the spectral radiancy per unit range of wavelength has its max^m value is given by

$$\boxed{\lambda_m T} = \frac{hc}{kx} = \frac{hc}{4.965K} = b$$

i.e $\boxed{\lambda_m T = b}$

This Wien's displacement law

* Solar Constant *

~~The Sun is a source of thermal energy and it emits continuously radiant energy in all directions in space. The earth receives only a part of this energy. A considerable portion of this incoming radiation or energy is lost by reflection~~